TRANSPORT MATRICES OF FINITE BEAMS RESTING
ON ELASTIC FOUNDATIONS
A SUMMARY BY
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ABSTRACT

A method of derivation of the coefficients of the transport matrix for a finite
element resting on one and two-parameter elastic foundation soils is outlined. The
coefficients are summarized for both the Benoulli–Euler and the Timoshenko beam
model in tabular form, for general reference. In addition, bounds on the parameters
describing beam/soil flexure, beam and soil shear, which govern the validity of the
two-parameter elastic solutions are provided in graphical form.

INTRODUCTION

Through the availability of powerful micro–computers nowadays available to a large
number of design engineers, the application of finite element technique for use in
foundation engineering design is becoming increasingly more feasible. A large
number of authors [1,2,3,4,10], among many, have formulated solutions to this type
of problem. Scott[11] and Selvadurais[12] in their books provide detailed reviews on
various models published during the past. Without exceptions, all feasible,
numerical solutions on elastic beams resting on elastic foundation soils are based on
some system of linear springs representing the foundation soil in question. The
majority of solutions to problems on elastic foundations are based on the classical
Bernalli – Euler beam theory wherein the effect of beam shear is neglected. This
type of beam has been modelled as being supported by a soil exhibiting zero shear
resistance or one that exhibits non-zero shear resistance. The most practical and,
thus, the most popular two beam foundation types are generally referred to as
"one-parameter elastic foundation" and "two-parameter elastic foundation" both
based on linear elastic spring analogy. Since, under certain conditions, neglecting
beam shear affects were recognized to lead to significant errors, researchers have
developed solutions to the problem of concern herein based on the Timoshenko
beam resting on one or two-parameter elastic foundations.

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This paper deals with the transport matrix which in structural engineering has been subject to treatment already back in the sixties [7,8,9]. Filipkowski[5], in context with the development of a new version of displacement method, known as exact finite element method, showed that the transport matrix bears a link to the force-displacement relationship of a finite element. With this fact in mind the transport matrix for a finite element of the Timoshenko type, resting on a two-parameter elastic foundation, was formulated. It is the objective of this paper to outline the development of the transport matrix pertaining to a finite Timoshenko beam element resting on a one or two-parameter elastic spring foundation, to list the matrix coefficients for the Bernoulli–Euler and Timoshenko beam models and to indicate bounds of parameters governing the validity of the two-parameter elastic foundation solution. The development of the force-displacement relationship for the Timoshenko beam element resting on a two-parameter elastic spring foundation is subject to a detailed presentation in another paper which is currently under preparation.

GOVERNING DIFFERENTIAL EQUATIONS OF TIMOSHENKO BEAM

The Timoshenko beam theory [6] includes the effect of transverse shear. It is assumed that (a) lateral deflections are small when compared with the thickness of the beam, (b) planes normal to the neutral axis remain plane but do not, in general, remain normal to the neutral axis and (c) stresses transverse to the beam axis are negligible. Consider Figure 1 below:

![Kinematics of Deformation of the Beam Element](image)

Fig. 1 Kinematics of Deformation of the Beam Element.

from where it can be seen that the angular bending distortion, \( \phi \), or rotation of the normal to the neutral axis of an element \( dx \) is of the form:

\[
\phi = \frac{dv}{dx} - \alpha \tag{1}
\]
in which \( \frac{dv}{dx} \) stands for the slope of the elastic curve of the beam element at point \( x \) and \( \bar{\alpha} \) represents the additional rotation of the neutral beam axis due to transverse shear deformation.

If \( k \) denotes the modulus of subgrade reaction and \( k_1 \) is a measure of rotational stiffness of the subgrade then, by making reference to Figure 1, the total potential energy of the element – force system, can be written:

\[
\delta I(v,\phi) = \int_0^1 EI(\phi')^2 dx + \int_0^1 \kappa G(v',\phi')^2 dx + \int_0^1 kv^2 dx + \int_0^1 k\phi dx - \int_0^1 p(x)v dx - T\phi|_0^1 - M\phi|_0^1,
\]

where:
- \( EI \) = flexural rigidity of the beam
- \( GA \) = shear rigidity of the beam
- \( I \) = moment of inertia of the cross-section
- \( A \) = area of cross-section
- \( l \) = element length
- \( x \) = coordinate, independent variable along neutral beam axis
- \( \phi(x) \) = rotation of the normal to the neutral axis
- \( v(x) \) = vertical displacement of neutral axis, \( \phi(x) \)
- \( \phi \) = first derivative
- \( \kappa \) = warping constant
- \( p(x) \) = external, continuous load normal to the neutral axis
- \( F(x) \) = section shear forces in \( y \) – direction
- \( M(x) \) = section bending moments.

The application of the principle of virtual work, from which it follows that equilibrium of a deformable system is subject to the condition:

\[
\delta I(v,\phi) = 0,
\]

where \( \delta \) denotes the first variation, leads to the following set of two simultaneous differential equations in \( v \) and \( \phi \) for beam elements with constant prismatic cross-sections,

\[
-\kappa G A \frac{d^2 v}{dx^2} + kv + \kappa G A \frac{d\phi}{dx} = p(x)
\]

\[
\kappa G A \frac{dv}{dx} + EI \frac{d^2 \phi}{dx^2} - (\kappa G A + k_1)\phi = 0.
\]
In the absence of geometric boundary conditions, the associated natural boundary conditions are:

\[ M(0) = EI\phi'(0) \text{ and } T(0) = RGA[\nu'(0) - \phi(0)] \quad (5) \]

By assuming a function \( f(x) \) such that

\[ \nu = -\frac{1}{RGA}[EI\frac{d^2}{dx^2} - (RGA + k_1)]f(x); \quad \phi = \frac{df(x)}{dx} \quad (6) \]

and by substitution of the expressions in Eq. (6) into Eq. (4), one obtains the governing differential equation of a Timoshenko beam resting on a two-parameter elastic foundation:

\[ EI\frac{d^4f(x)}{dx^4} - (k_1 + \frac{kE}{RGA})\frac{d^2f(x)}{dx^2} + k(1 + \frac{k_1}{RGA})f(x) = p(x) \quad (7) \]

with \( \eta = 1/\sqrt{GA} \), Eq. (6) can be rewritten to read:

\[ v(x) = -\frac{1}{\sqrt{G}}\eta\frac{d^2f(x)}{dx^2} + (1 + k_1\eta)f(x), \quad \phi(x) = \frac{df(x)}{dx} \quad (8) \]

According to Eq. (5) and (6) the bending moment, \( M \), and the shear force, \( T \), are given by Eq. (9) below:

\[ M(x) = EI\frac{d^2f(x)}{dx^2} \quad \text{and} \quad T(x) = -EI\frac{d^2f(x)}{dx^2} + k_1\frac{df(x)}{dx} \quad (9) \]

**SOLUTION TO HOMOGENEOUS DIFFERENTIAL EQUATION**

A solution to the homogeneous differential equation (7) \([p(x) = 0]\) does exist in the form of

\[ f^{(n)}(x) = A\phi_1^{(n)}(x) + B\phi_2^{(n)}(x) + C\phi_3^{(n)}(x) + D\phi_4^{(n)}(x) \quad (10) \]

where \((n)\) stands for the \(n\)th derivative \((n \geq 0)\); \(A, B, C\) and \(D\) are constants of integration and the functions \(\phi_m(x)\) take the form,

\[ \begin{align*}
\phi_1(x) &= \sin(a\lambda x) \cdot \sinh(b\lambda x) \\
\phi_2(x) &= \sin(a\lambda x) \cdot \cosh(b\lambda x) \\
\phi_3(x) &= \cos(a\lambda x) \cdot \sinh(b\lambda x) \\
\phi_4(x) &= \cos(a\lambda x) \cdot \cosh(b\lambda x).
\end{align*} \quad (11) \]
The coefficients $a$, $b$ and $\lambda$ are defined:

$$a = \sin \frac{\varphi}{2}; \quad b = \cos \frac{\varphi}{2}; \quad \lambda = \left[ \frac{k}{EI(1+k_1\eta)} \right]^{1/4}$$

where $\varphi$ is determined from the relationship,

$$\varphi = \arctan \left[ \frac{4kEI(1+k_1\eta) - (k_1+kEI\eta)^2}{(k_1+kEI\eta)} \right]^{1/2}$$

By defining

$$k_2 = \lambda^2EI = [kEI(1+k_1\eta)]^{1/2}$$

and making use of TABLE I below, one can readily express the values at $x = 0$ of the functions given in Eq. (8) and (9) in matrix form:

$$\begin{bmatrix} v(0) \\ \phi(0) \\ M(0) \\ T(0) \end{bmatrix} = \begin{bmatrix} -2abk_2\eta \\ 0 \\ 2abk_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a\lambda \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ b\lambda \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1+k_1\eta+(a^2-b^2)k_2\eta \\ (a^2-b^2)k_2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

By inspection of Eq. (15), it is immediately apparent that the following systems of linear equations can be written and solved in two sets of two simultaneous linear equations to obtain the constants of integration $(A,B,C,D)$ in terms of the boundary parameters shown in Fig. 2(a) and 2(b).
\[
\begin{bmatrix}
-2abk_2\eta & 1+k_1\eta+(a^2-b^2)k_2\eta \\
2abk_2 & -(a^2-b^2)k_2 \\
\end{bmatrix}
\begin{bmatrix}
A \\
D \\
\end{bmatrix}
= \begin{bmatrix}
v(0) \\
M(0) \\
\end{bmatrix}
= \begin{bmatrix}
v_i \\
-M_i \\
\end{bmatrix}
\quad (16(a))
\]

and

\[
\begin{bmatrix}
a\lambda \\
b\lambda \\
\end{bmatrix}
\begin{bmatrix}
\lambda^{1+k_1\eta+(a^2-b^2)k_2\eta} \\
\lambda^{-1+k_1\eta-(1-4a^2)k_2} \\
\end{bmatrix}
\begin{bmatrix}
B \\
C \\
\end{bmatrix}
= \begin{bmatrix}
\phi(0) \\
T(0) \\
\end{bmatrix}
= \begin{bmatrix}
\phi_i \\
-T_i \\
\end{bmatrix}
\quad (16(b))
\]

Fig. 2. Boundary Parameters for Beam Element.

Employing Cramer's rule and by making reference to the initial parameters of Figure 2 (b), the constants of integration turn out to be of the form:

\[
A = \frac{a^2-b^2}{2ab(1+k_1\eta)} v_i - \frac{1}{2ab(1+k_1\eta)k_2} M_i,
\]

\[
B = \frac{k_1(1-4a^2)k_2}{2abk_2} \phi_i + \frac{1}{2abk_2} T_i,
\]

\[
C = -\frac{k_1(1-4b^2)k_2}{2bk_2} \phi_i - \frac{1}{2bk_2} T_i,
\]

\[
D = \frac{1}{1+k_1\eta} v_i - \frac{\eta}{1+k_1\eta} M_i.
\]

Making use of the constants in Eq. (17), the tabulated functions of TABLE II below in combination with Eq. (11) and Eq. (10), permits one to write the solution vector \(\{v(x), \phi(x), T(x), M(x)\}^T\) in matrix form with reference to Eq. (8) and (9):

\[
\begin{bmatrix}
v(x) \\
\phi(x) \\
T(x) \\
M(x) \\
\end{bmatrix}
= \begin{bmatrix}
B_{VV} & B_{V\phi} & B_{VT} & B_{VM} \\
B_{QV} & B_{Q\phi} & B_{QT} & B_{QM} \\
B_{TV} & B_{T\phi} & B_{TT} & B_{TM} \\
B_{MV} & B_{M\phi} & B_{MT} & B_{MM} \\
\end{bmatrix}
= \begin{bmatrix}
v_i \\
\phi_i \\
T_i \\
M_i \\
\end{bmatrix}
\quad (18)
\]

Uhandisi Journal Vol. 14 No. 1 1990
TABLE II. Function $\phi_m(x)$ and Its Derivatives For Arbitrary Values of $x$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\phi_m$</th>
<th>$\phi'_m$</th>
<th>$\phi''_m$</th>
<th>$\phi'''_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\phi_1$</td>
<td>$\lambda(a_2b_3 + b_0\phi_2)$</td>
<td>$-\lambda^2(a^2-b^2)\phi_1 + 2ab\lambda \phi_4$</td>
<td>$\lambda^3b(1-4a^2\phi_1 - \lambda^3a(1-4b^2)\phi_2$</td>
</tr>
<tr>
<td>2</td>
<td>$\phi_2$</td>
<td>$\lambda(a_2b_4 + b_1\phi_1)$</td>
<td>$-\lambda^2(a^2-b^2)\phi_2 + 2ab\lambda \phi_3$</td>
<td>$\lambda^3b(1-4a^2)\phi_1 - \lambda^3a(1-4b^2)\phi_4$</td>
</tr>
<tr>
<td>3</td>
<td>$\phi_3$</td>
<td>$\lambda(-a_1\phi_1 + b_3\phi_4)$</td>
<td>$-\lambda^2(a^2-b^2)\phi_3 - 2ab\lambda \phi_2$</td>
<td>$\lambda^3b(1-4a^2)\phi_4 + \lambda^3a(1-4b^2)\phi_1$</td>
</tr>
<tr>
<td>4</td>
<td>$\phi_4$</td>
<td>$\lambda(-a_2\phi_2 + b_2\phi_3)$</td>
<td>$-\lambda^2(a^2-b^2)\phi_4 - 2ab\lambda \phi_1$</td>
<td>$\lambda^3b(1-4a^2)\phi_3 + \lambda^3a(1-4b^2)\phi_2$</td>
</tr>
</tbody>
</table>

To provide an insight into how the elements of the transport matrix are calculated, the first of Eq. (8) is considered. This results in the first row of the elements in the matrix, $[B(x)]$, shown in Eq. (18):

$$v(x) = -EI\eta\left[A\phi'_1(x) + B\phi'_2(x) + C\phi'_3(x) + D\phi'_4(x)\right]$$

$$+ (1 + k_1\eta)\left[A\phi_1(x) + B\phi_2(x) + C\phi_3(x) + D\phi_4(x)\right]$$

(19)

with reference to TABLE II and Eq. (17), one may write

$$v(x) = \left[\frac{a^2-b^2}{2ab(1+k_1\eta)}v_i - \frac{1 + k_1(\lambda_1 - \lambda_2)}{2ab(1+k_1\eta)}\lambda_1 T_i\right]$$

$$\times \left[\lambda^2 E_1 n(a^2-b^2)\eta M_1 - 2\lambda^2 E_1 n \eta a b \phi_4 \right]$$

$$- (1 + k_1\eta)\phi_i$$

$$+ \left[\frac{k_1(1-4a^2)}{2a\lambda_k} \phi_i + \frac{1}{2a\lambda_k} T_i\right]$$

$$\times \left[\lambda^2 E_1 n(a^2-b^2)\phi_2 - 2\lambda^2 E_1 n a b \phi_3 + (1 + k_1\eta)\phi_2\right]$$

$$- \left[\frac{k_1(1-4b^2)}{2b\lambda_k} \phi_i + \frac{1}{2b\lambda_k} T_i\right]$$

$$\times \left[\lambda^2 E_1 n(a^2-b^2)\phi_3 + 2\lambda^2 E_1 n a b \phi_2 + (1 + k_1\eta)\phi_3\right]$$

$$+ \left[\frac{1}{1+k_1\eta} v_i - \frac{\eta M_1}{1+k_1\eta}\right]$$

$$\times \left[\lambda^2 E_1 n(a^2-b^2)\phi_4 + 2\lambda^2 E_1 n a b \phi_1 + (1 + k_1\eta)\phi_4\right]$$

(20)

If all terms in Eq. (20) containing $v_i$ are summed up and recognition of the fact that $a = \sin(\varphi/2)$, $b = \cos(\varphi/2)$ and $a^2 + b^2 = 1$ is taken, $B_{vv}$ is found to be of the form:

$$B_{vv} = \frac{(1+k_1\eta)(a^2-b^2) + k_2\eta}{2ab(1+k_1\eta)} (x) + \phi_4(x)$$

(21)

Uhambisi Journal Vol. 14 No.1 1990
Similarly, all terms containing $\phi_i$ are collected together to obtain $B_{\nu \phi}$. Other elements of the transport matrix are calculated in the same way by considering each row with the corresponding equations.

**TRANSPORT MATRIX COEFFICIENTS**

Transport matrix coefficients are listed in the APPENDIX for both Timoshenko and Bernoulli–Euler beams resting on one and two-parameter elastic foundations.

**TABLE IV** contains all elements of the transport matrix for a Timoshenko beam resting on a two-parameter elastic foundation. For the Bernoulli–Euler beam resting on a two-parameter elastic foundation, one assumes $\eta = 0$ and obtains all elements of the transport matrix as shown in **TABLE V**. The one-parameter transport matrix coefficients subject to condition of $k_1 = 0$ are listed in **TABLE VI** for the Timoshenko beam element whereas those for the Bernoulli–Euler beam are tabulated in **TABLE VII** (note: both $\eta$ and $k_1$ are zero). It should be born in mind that all $\phi_m = \phi_m(x)$. Also, note that the coefficients $a$ and $b$ used in tables bear no relationship with the constants of integration in Eq. (17). The coefficients $a$ and $b$ are defined by Eq. (12) and (13).

**PARAMETER RANGE OF APPLICABILITY**

For the solution of the homogeneous differential equation for the Timoshenko beam element resting on a two-parameter elastic foundation to be valid, the following condition must hold,

$$k_1 + kE\eta \leq \sqrt{4kE(1+k_1\eta)}$$  \hspace{1cm} (22)

If one defines the dimensionless quantities $\Lambda$, $\Gamma_1$ and $\Gamma_2$ which represent the influence of the first foundation parameter $k$, second foundation parameter $k_1$ and the effect of transverse beam shear, $\eta$, respectively, where

$$\Lambda = 4 \frac{k l^4}{EI}, \quad \Gamma_1 = \frac{k_1 l^2}{EI} \quad \text{and} \quad \Gamma_2 = \frac{E l^3}{EI}$$  \hspace{1cm} (23)

Eq. (22) can be rewritten to read:

$$4(1 + \Gamma_1 \Gamma_2)\Lambda^4 \geq (\Gamma_1 + \Gamma_2 \Lambda^4)^2.$$  \hspace{1cm} (24)
It should be kept in mind that $\Gamma_1$ cannot exist when $\Lambda = 0$ from physical point of view. For the second foundation parameter, $k_1$, to have a meaning, vertical support of the elastic foundation, which is defined by $k$, must exist.

By inspection of Eq. (24) above, it becomes evident that three quantities determine the condition. The authors determined the range of $\Gamma_1$ for different values of $\Gamma_2$ and $\Lambda$ and obtained the relationship shown in Fig. 5(a) through 5(e). Fig. 5(a) represents the case $\Gamma_2 = 0$, which corresponds to the Bernoulli–Euler beam resting on an elastic foundation. It should be noted that the range of $\Gamma_1$ increases with increasing values of $\Lambda$.

For given values of $\Gamma_1$ and $\Gamma_2$, one can also find the range of $\Lambda$ for which the condition in Eq. (24) can be satisfied. These are shown in TABLE III in which case the range of $\Lambda$ decreases with the increase in the values of $\Gamma_2$, for any given value of $\Gamma_1$.

TABLE III. Range of $\Lambda$ For Given Values of $\Gamma_1$ and $\Gamma_2$

<table>
<thead>
<tr>
<th>$\Gamma_1$</th>
<th>$\Gamma_2$</th>
<th>Lower Limit of $\Lambda$</th>
<th>Upper Limit of $\Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.005</td>
<td>0.7067</td>
<td>20.0124</td>
</tr>
<tr>
<td></td>
<td>0.010</td>
<td>0.7063</td>
<td>14.1597</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>0.7050</td>
<td>8.9720</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>0.7028</td>
<td>6.3634</td>
</tr>
<tr>
<td>2.5</td>
<td>0.005</td>
<td>1.1163</td>
<td>20.0311</td>
</tr>
<tr>
<td></td>
<td>0.010</td>
<td>1.1146</td>
<td>14.1850</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>1.1096</td>
<td>9.0128</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>1.1015</td>
<td>6.4197</td>
</tr>
<tr>
<td>5.0</td>
<td>0.005</td>
<td>1.5763</td>
<td>20.0620</td>
</tr>
<tr>
<td></td>
<td>0.010</td>
<td>1.5715</td>
<td>14.2791</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>1.5577</td>
<td>9.0788</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>1.5365</td>
<td>6.5085</td>
</tr>
<tr>
<td>7.5</td>
<td>0.005</td>
<td>1.9276</td>
<td>20.0926</td>
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<tr>
<td></td>
<td>0.010</td>
<td>1.9180</td>
<td>14.2770</td>
</tr>
<tr>
<td></td>
<td>0.025</td>
<td>1.8945</td>
<td>9.1427</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>1.8580</td>
<td>6.5918</td>
</tr>
</tbody>
</table>
\[ \Lambda = \left( \frac{kl^4}{EI} \right)^{1/4} \]

\[ \Gamma_1 = \frac{k_1l^2}{EI} \]

\[ \Gamma_2 = \frac{EI}{l^2} \eta \]

Fig. 5 Upper and Lower Limits of $\Gamma_1$ versus $\Lambda$
CONCLUSIONS AND RECOMMENDATIONS

Based on the study presented in this paper, the following conclusions and recommendations are set forth.

1. The use of finite element technique in foundation engineering design is becoming increasingly more feasible due to the widespread availability of powerful micro-computers. Thus, the transport matrix gains an important role since it is a key element in the formulation of a finite element solution.

2. The transport matrix provides complete information on the behaviour of the beam at any point \( x \) in the range \( x_i \leq x \leq x_k \) to the right of the element boundary, \( x_i \), provided the information at \( x_i(x = 0) \) represented by the column vector \( \{v(0), \phi(0), T(0), M(0)\}^T \) is known.

3. Since the determination of the transport matrix coefficients is quite work-involved and the nature of the transport matrices for the problem of concern in this paper have general validity, transport matrix coefficients were tabulated in TABLES IV through VIII for both Timoshenko and Bernoulli-Euler type of beam models. These coefficients may be used directly in the development of a stiffness matrix for a given beam/foundation soil system if in compliance with the assumptions made in the choice of a particular model.

4. The results shown in Figure 5 and Table III serve as a guidance for the setting up of proper constants in a given problem of a beam resting on an elastic spring foundation. These constitute the computer input for a particular numerical solution sought by the engineer.

REFERENCES


LIST OF SYMBOLS

- \( A \) = Area of Beam Cross-Section
- \( G \) = Shear Modulus of Beam Material
- \( l(\phi, v) \) = Total Potential Energy
- \( M(x) \) = Section Bending Moment
- \( M_i \) = \(-M(0)\) Applied End Moment
\( M_k \) = \(-M(l)\) Applied End Moment
\( p(x) \) = External, Continuous Load
\( v(x) \) = Total Vertical Displacement of Neutral Axis
\( \phi \) = First Derivative
\( \phi_m(x) \) = Rotation of Neutral Axis Due to Bending
\( \phi_m(x) \) = Arbitrary Functions for \( m = 1, 2, 3, 4 \)
\( E \) = Young's Modulus of Beam Material
\( I \) = Area Moment of Inertia of Cross-Section
\( T(x) \) = Section Shear Force in y-direction
\( T_s \) = \(-T(0)\) Applied End Force
\( T_k \) = \(-M(l)\) Applied End Force
\( k, k_i \) = First and Second Soil Parameters
\( l \) = Beam Element Length
\( x \) = Coordinate, Independent Variable Along Neutral Beam Axis
\( N \) = Warping Constant
\( \delta \) = First Variation
\( \Gamma_1, \Gamma_2, \Lambda \) = Dimensionless Parameters
APPENDIX
TRANSPORT MATRIX COEFFICIENTS
TABLE IV: Timoshenko Beam / Two-Parameter Elastic Foundation

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = \frac{4(1+k_1\eta)k_1}{4EI}$, $k_2 = \lambda^2 EI$, $A = 1+\eta(k_1+k_2)$, $B = 1+\eta(k_1-k_2)$</td>
<td></td>
</tr>
<tr>
<td>$B_{v\nu}$</td>
<td>$\frac{a^2A-b^2B}{ab(A+B)} \cdot \phi_1 + \phi_4$</td>
</tr>
<tr>
<td>$B_{\phi\nu}$</td>
<td>$-\frac{\lambda}{ab(A+B)} \cdot (b\phi_2 + a\phi_3)$</td>
</tr>
<tr>
<td>$B_{T\nu}$</td>
<td>$\frac{\lambda}{ab(A+B)} [-b(k_1-k_2)\phi_2 + a(k_1+k_2)\phi_3]$</td>
</tr>
<tr>
<td>$B_{\phi\nu}$</td>
<td>$\frac{k_2}{ab(A+B)} \cdot \phi_1$</td>
</tr>
<tr>
<td>$B_{v\phi}$</td>
<td>$\frac{1}{2ab\lambda} \cdot (b\phi_2 + a\phi_3)$</td>
</tr>
<tr>
<td>$B_{\phi\phi}$</td>
<td>$\frac{a^2(k_1+k_2)+b^2(k_1-k_2)}{2abk_2} \cdot \phi_1 + \phi_4$</td>
</tr>
<tr>
<td>$B_{T\phi}$</td>
<td>$\frac{a^2(k_1+k_2)^2+b^2(k_1-k_2)^2}{2abk_2^2} \cdot \phi_1$</td>
</tr>
<tr>
<td>$B_{\phi\phi}$</td>
<td>$\frac{1}{2ab\lambda} \cdot [b(k_1-k_2)\phi_2 + a(k_1+k_2)\phi_3]$</td>
</tr>
<tr>
<td>$B_{vT}$</td>
<td>$\frac{1}{2ab\lambda k_2} \cdot (bB\phi_2 - aA\phi_3)$</td>
</tr>
<tr>
<td>$B_{\phi T}$</td>
<td>$\frac{1}{2abk_2} \cdot \phi_1$</td>
</tr>
<tr>
<td>$B_{TT}$</td>
<td>$\frac{a^2(k_1+k_2)+b^2(k_1-k_2)}{2abk_2} \cdot \phi_1 - \phi_4$</td>
</tr>
<tr>
<td>$B_{\phi\phi}$</td>
<td>$\frac{1}{2ab\lambda} \cdot (b\phi_2 + a\phi_3)$</td>
</tr>
<tr>
<td>$B_{v\phi}$</td>
<td>$\frac{-1}{2abk_2} \cdot \phi_1$</td>
</tr>
<tr>
<td>$B_{\phi\phi}$</td>
<td>$\frac{-\lambda}{ab(A+B)k_2} \cdot (bB\phi_2 + aA\phi_3)$</td>
</tr>
<tr>
<td>$B_{T\phi}$</td>
<td>$\frac{-\lambda}{ab(A+B)} \cdot (b\phi_2 - a\phi_3)$</td>
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<tr>
<td>$B_{\phi\phi}$</td>
<td>$\frac{a^2A-b^2B}{ab(A+B)} \cdot \phi_1 - \phi_4$</td>
</tr>
<tr>
<td>( B_{vv} )</td>
<td>( \frac{a^2-b^2}{2ab} \phi_1 + \phi_4 )</td>
</tr>
<tr>
<td>( B_{\phi v} )</td>
<td>( -\frac{\lambda}{2ab} (b\phi_2 - a\phi_3) )</td>
</tr>
<tr>
<td>( B_{TV} )</td>
<td>( \frac{\lambda}{2ab} [-b(k_1-k_2)\phi_2 + a(k_1+k_2)\phi_3] )</td>
</tr>
<tr>
<td>( B_{\phi v} )</td>
<td>( \frac{-k_2}{2ab} \phi_1 )</td>
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<tr>
<td>( B_{v\phi} )</td>
<td>( \frac{1}{2ab\lambda} (b\phi_2 + a\phi_3) )</td>
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<tr>
<td>( B_{\phi \phi} )</td>
<td>( \frac{a^2(k_1+k_2) + b^2(k_1-k_2)}{2abk_2} \phi_1 + \phi_4 )</td>
</tr>
<tr>
<td>( B_{T\phi} )</td>
<td>( \frac{a^2(k_1+k_2)^2 + b^2(k_1-k_2)^2}{2abk_2} \phi_1 )</td>
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<tr>
<td>( B_{\phi \phi} )</td>
<td>( \frac{1}{2ab\lambda} [-b(k_1-k_2)\phi_2 + a(k_1+k_2)\phi_3] )</td>
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<tr>
<td>( B_{vt} )</td>
<td>( \frac{1}{2ab\lambda} (b\phi_2 - a\phi_3) )</td>
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<tr>
<td>( B_{\phi t} )</td>
<td>( \frac{1}{2abk_2} \phi_1 )</td>
</tr>
<tr>
<td>( B_{TT} )</td>
<td>( \frac{a^2(k_1+k_2) + b^2(k_1-k_2)}{2abk_2} \phi_1 - \phi_4 )</td>
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<tr>
<td>( B_{MT} )</td>
<td>( \frac{1}{2ab\lambda} (b\phi_2 + a\phi_3) )</td>
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<tr>
<td>( B_{v\phi} )</td>
<td>( \frac{-1}{2abk_2} \phi_1 )</td>
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<tr>
<td>( B_{\phi \phi} )</td>
<td>( \frac{-\lambda}{2abk_2} (b\phi_2 + a\phi_3) )</td>
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<tr>
<td>( B_{TM} )</td>
<td>( \frac{-\lambda}{2ab} (b\phi_2 - a\phi_3) )</td>
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<tr>
<td>( B_{\phi \phi} )</td>
<td>( \frac{a^2-b^2}{2ab} \phi_1 - \phi_4 )</td>
</tr>
<tr>
<td>$B_{vv}$</td>
<td>$\frac{a^2}{2ab} - b^2 + k_2\eta \phi_1 + \phi_4$</td>
</tr>
<tr>
<td>$B_{\phi v}$</td>
<td>$\frac{\lambda}{2ab} - (b\phi_2 - a\phi_3)$</td>
</tr>
<tr>
<td>$B_{Tv}$</td>
<td>$\frac{\lambda}{2ab} - (b\phi_2 + a\phi_3)$</td>
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<tr>
<td>$B_{\phi v}$</td>
<td>$-\frac{k_2}{2ab} \phi_1$</td>
</tr>
<tr>
<td>$B_{v\phi}$</td>
<td>$-\frac{1}{2ab\lambda} - (b\phi_2 + a\phi_3)$</td>
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<tr>
<td>$B_{\phi \phi}$</td>
<td>$\frac{a^2}{2ab} - b^2 + \phi_1 + \phi_4$</td>
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<tr>
<td>$B_{T\phi}$</td>
<td>$\frac{k_2}{2ab} \phi_1$</td>
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<tr>
<td>$B_{\phi T}$</td>
<td>$\frac{k_2}{2ab\lambda} - (b\phi_2 + a\phi_3)$</td>
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<tr>
<td>$B_{vT}$</td>
<td>$\frac{1}{2ab\lambda k_2} [b(1-k_2\eta)\phi_2 - a(1+k_2\eta)\phi_3]$</td>
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<tr>
<td>$B_{\phi T}$</td>
<td>$\frac{1}{2abk_2} \phi_1$</td>
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<td>$B_{TT}$</td>
<td>$\frac{a^2}{2abk_2} \phi_1 + \phi_4$</td>
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<td>$B_{\phi T}$</td>
<td>$\frac{1}{2ab\lambda} - (b\phi_2 + a\phi_3)$</td>
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<td>$B_{vH}$</td>
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<tr>
<td>$B_{\phi H}$</td>
<td>$\frac{-\lambda}{2abk_2} [b(1-k_2\eta)\phi_2 + a(1+k_2\eta)\phi_3]$</td>
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<tr>
<td>$B_{T\phi}$</td>
<td>$\frac{-\lambda}{2ab} - (b\phi_2 - a\phi_3)$</td>
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<td>$B_{\phi H}$</td>
<td>$\frac{a^2}{2ab} - b^2 + k_2\eta \phi_1 - \phi_4$</td>
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<tr>
<td>$B_{vv}$</td>
<td>$\phi_4$</td>
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<tr>
<td>$B_{\phi v}$</td>
<td>$-\frac{k}{\sqrt{2}}(\phi_2 - \phi_3)$</td>
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<tr>
<td>$B_{Tv}$</td>
<td>$-\frac{k}{\sqrt{2}}(\phi_2 + \phi_3)$</td>
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<tr>
<td>$B_{\phi T}$</td>
<td>$-k_2 \cdot \phi_1$</td>
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<tr>
<td>$B_{v\phi}$</td>
<td>$\frac{1}{\lambda k_{24}^2}(\phi_2 + \phi_3)$</td>
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<td>$\phi_4$</td>
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<td>$k_2 \cdot \phi_1$</td>
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<td>$B_{m \phi}$</td>
<td>$-\frac{k_2}{\lambda k_{24}^2}(\phi_2 - \phi_3)$</td>
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<td>$\frac{1}{\lambda k_{24}^2}(\phi_2 - \phi_3)$</td>
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<tr>
<td>$B_{\phi T}$</td>
<td>$\frac{1}{k_2} \cdot \phi_1$</td>
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<td>$B_{TT}$</td>
<td>$- \phi_4$</td>
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<tr>
<td>$B_{n T}$</td>
<td>$\frac{1}{\lambda k_{24}^2}(\phi_2 + \phi_3)$</td>
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<tr>
<td>$B_{v n}$</td>
<td>$-\frac{1}{k_2} \cdot \phi_1$</td>
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<tr>
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<td>$-\frac{\lambda}{k_{24}^2 k_2}(\phi_2 + \phi_3)$</td>
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<tr>
<td>$B_{T n}$</td>
<td>$-\frac{\lambda}{k_{24}^2}(\phi_2 - \phi_3)$</td>
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<tr>
<td>$B_{n n}$</td>
<td>$- \phi_4$</td>
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