EXTRACTING CONSISTENT WEIGHT RATIO MATRICES FROM INCONSISTENT JUDGEMENT MATRICES.

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Abstract:

The Analytic Hierarchy Process (AHP) developed by Saaty more than a decade ago has been widely applied in diverse decision making problems ranging from socio-economic planning to engineering designs. One of the basic steps of this popular decision support tool is concerned with estimating a ratio scale of a set of entities from their inconsistent pairwise comparison judgement matrix.

A number of estimation procedures for this purpose have been proposed. These include Saaty's eigenvector method, various geometric means methods, and methods using logarithmic least squares, least squares, and means transformations, amongst others.

One thing common to all these methods is the assumption that there exists a consistent weight ratio matrix whose entries are the ratios of the components of a ratio scale being estimated.

This paper describes a procedure for extracting this consistent weight ratio matrix from an inconsistent pairwise comparison judgement matrix.

Introduction.

The Analytic Hierarchy Process: A Review

The Analytic Hierarchy Process (AHP) developed by Thomas Saaty [1] is a multicriteria decision making technique which decomposes a complex problem into hierarchy, in which each level is composed of specific elements. The overall objective of the decision lies at the top of the hierarchy, and the criteria, sub criteria and decision alternatives are on descending levels of this hierarchy. The hierarchy does not need to be complete, i.e., an element in a given level does not have to function as a criteria for all the elements in the level below. Thus a hierarchy can be divided into sub hierarchies sharing only a common topmost element.

Once the hierarchical model has been structured for the problem, the participating decision makers provide pairwise comparisons for each level of the hierarchy in order to obtain the weight factor of each element on that level with respect to one element in the next higher level. This weight factor provides a measure of the relative importance of this element for the decision maker.

To compute the weight factors of n elements, the input consists of comparing each pair of the elements using a scale set of

$$S = \left\{ \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9 \right\}$$

The pairwise comparison of element i and element j, with respect to an element in the next higher level, is placed in the position of a_{ij} of the pairwise comparison matrix A as shown below:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The reciprocal value of this comparison is placed in the position a_{ji} of A in order to preserve consistency of judgement. Thus, given n elements, the participating decision maker compares the relative importance of one element with respect to a second element, using the 9-point scale shown in Table 1.
Hence, if element one was strongly favoured over element two, for example, then $a_{12} = 5$. If the converse was true, element two was strongly favoured over element one, a_{12} is the reciprocal value $1/5$. The pairwise comparison matrix is called a reciprocal matrix for obvious reasons.

Table 1: The 9-point scale for pairwise comparisons

Importance	Definition	Explanation
1	Equal importance	Two elements contribute identically to the objective
3	Weak dominance	Experience or judgement slightly favours one element over another.
5	Strong dominance	Experience or judgement strongly favours one element over another.
7	Demonstrated dominance	An element's dominance is demonstrated in practice.
9	Absolute dominance	The evidence favouring an element over another is affirmed to the highest possible order.
2,4,6,8	Intermediate values	Further subdivision or compromise is needed.

The Mathematical Basis of the Analytic Hierarchy Process.

The intuition behind the Analytic Hierarchy Process is that in a perfect world, the pairwise comparison matrix A would be identical to the matrix

$$X = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & & & \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix}$$

where w_i is the relative weight of element i.

Various methods have been proposed to extract values $\{w_i\}$ from the matrix A, which would be close approximates to the values in X. Saaty [2] proposed the use of the dominant right eigenvector of **A** as an estimator of the ratio scale of the set of entities in a hierarchy level whose inconsistent judgement matrix is **A**.

Crawford and Williams [3] suggested the row geometric means vector on the basis of statistical and logarithmic least squares considerations. Their suggestion has been supported by Barzilai et al [4,5,6] who showed that, when additive normalisation is replaced by multiplicative normalisation, the method satisfies basic axioms of consistency.

Other proposed estimator methods include least squares [7], the harmonic mean or left eigenvector [8], the mean transformation [9] and renormalisation after the estimation of ratios [10,11].

One thing common to all these estimation methods is the assumption that there exists a consistent weight ratio matrix whose entries are the ratios of the components of the ratio scale being estimated. The judgement matrix provided by the decision maker is accepted or rejected depending on how close it is to this consistent weight ratio matrix [12]. The advantage of extracting this consistency weight ratio matrix is that any normalised column yields the required ratio scale. This may reduce the computational burden involved in extracting values $\{w_i\}$ from matrix **A**.

In the next section a description of the method for extracting this matrix from an inconsistent judgement matrix supplied by the decision maker is given. This is followed by an example illustrating the application of this method. The paper ends with a summary and proposals for further research.

Extracting a Consistent Weight Ratio Matrix From an Inconsistent Judgement Matrix:

The Method.

Assuming that the consistency of the judgement matrix **A**, supplied by a decision maker, is within acceptable limits.

Let $\hat{w}_i$ for i = 1, 2, ..., n, be the approximate weight of object i, i.e. an estimate of w_i, and $\hat{w}_{ij}$ for i, j = 1, 2, ..., n, be the implied weight of object i from a comparison of object i with object j, so that:

$$\hat{w}_{ij} = a_{ij} * \hat{w}_j \qquad (1)$$

Note that even if $\hat{w}_i$ were a correct estimate of w_i (i.e. $\hat{w}_i = w_i$ exactly), the $\hat{w}_{ij}$ would still include the inconsistencies inherent in a_{ij}.

From (1) and the definition of a_{ij} we see that:

$$\begin{bmatrix} \hat{w}_{11}/\hat{w}_1 & \hat{w}_{12}/\hat{w}_2 & \cdots & \hat{w}_{1n}/\hat{w}_n \\ \hat{w}_{21}/\hat{w}_1 & \hat{w}_{22}/\hat{w}_2 & \cdots & \hat{w}_{2n}/\hat{w}_n \\ \vdots & & & \\ \hat{w}_{n1}/\hat{w}_1 & \hat{w}_{n2}/\hat{w}_2 & \cdots & \hat{w}_{nn}/\hat{w}_n \end{bmatrix} \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \\ \hat{w}_n \end{bmatrix} = n \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \\ \hat{w}_n \end{bmatrix} \qquad (2)$$

This is because $\hat{w}_{ij}$ is an estimate of $\hat{w}_i$ obtained from comparing object i and object j.

Note: $\sum_{j=1}^{n} \hat{w}_{ij} = n\hat{w}_i$. This means that $\hat{w}_i = \frac{1}{n}\sum_{j=1}^{n} \hat{w}_{ij}$. Hence, a good estimate of the weights $\hat{w}_i$ is the simple arithmetic mean of the weights $\hat{w}_{ij}$ across each row. Unfortunately, at this point we do not know these weights and so can not calculate $\hat{w}_i$.

$$\text{Let } \hat{w}_{*j} = \left[\prod_{i=1}^{n} \hat{w}_{ij} \right]^{\frac{1}{n}} \tag{3}$$

The $\hat{w}_{*j}$ are the n different estimates of the geometric mean of all the implied weights $\hat{w}_{ij}$, one for each column. Apart from inconsistencies the $\hat{w}_{*j}$ should be all the same.

Then the vector of column geometric means of the matrix on the left hand side of (2) is:

$$\hat{w} = \left(\frac{\hat{w}_{*j}}{\hat{w}_j} \right) \text{ for } j = 1,\ 2,\ \ldots,\ n \tag{4}$$

Dividing the columns of the matrix on the left hand side of (2) componentwise by (4) gives the following equation:

$$\begin{bmatrix} \hat{w}_{11}/\hat{w}_{*1} & \hat{w}_{12}/\hat{w}_{*2} & \cdots & \hat{w}_{1n}/\hat{w}_{*n} \\ \hat{w}_{21}/\hat{w}_{*1} & \hat{w}_{22}/\hat{w}_{*2} & \cdots & \hat{w}_{nn}/\hat{w}_{*n} \\ \vdots & & & \\ \hat{w}_{n1}/\hat{w}_{*1} & \hat{w}_{n1}/\hat{w}_{*2} & \cdots & \hat{w}_{nn}/\hat{w}_{*n} \end{bmatrix} \begin{bmatrix} \hat{w}_{*1} \\ \hat{w}_{*2} \\ \vdots \\ \hat{w}_{*n} \end{bmatrix} = n \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \vdots \\ \hat{w}_n \end{bmatrix} \tag{5}$$

The effect of this division is to eliminate the denominator out of each row (see 2) and replace them by the $\hat{w}_{*j}$ which are approximately all the same.

$$\text{Let } \hat{w}_{i*} = \left[\prod_{j=1}^{n} \frac{\hat{w}_{ij}}{\hat{w}_{*j}} \right]^{\frac{1}{n}} \quad \text{for } i, j = 1, 2, \ldots, n \tag{6}$$

Then $\hat{\mathbf{w}} = (\hat{w}_{i*})$ is an estimate of the multiplicatively synthesised weights of objects i for i =1, 2, ..., n. If true values of the $\hat{w}_i$ exist, the vector $\hat{\mathbf{w}}$ provides these up to a common multiplier. The geometric mean of the $\hat{w}_{*j}$, call this term

$$\hat{w}_{**} = \left[\prod_{j=1}^{n} \hat{w}_{*j} \right]^{\frac{1}{n}} . \tag{7}$$

So the numerator of $\hat{w}_{i*}$ is the best estimate of the w_i derived from all the values in row i and the denominator of $\hat{w}_{i*}$ is $\hat{w}_{**}$. Hence, $\hat{w}_{i*} = \dfrac{\hat{w}_i}{\hat{w}_{**}}$

Dividing the rows of (5) componentwise by (7) yields the following equation:

$$\begin{bmatrix} \hat{w}_{11}/\hat{w}_{1*}\hat{w}_{*1} & \hat{w}_{12}/\hat{w}_{1*}\hat{w}_{*2} & \cdots & \hat{w}_{1n}/\hat{w}_{1*}\hat{w}_{*n} \\ \hat{w}_{21}/\hat{w}_{2*}\hat{w}_{*1} & \hat{w}_{22}/\hat{w}_{2*}\hat{w}_{*2} & \cdots & \hat{w}_{2n}/\hat{w}_{2*}\hat{w}_{*n} \\ \vdots & & & \\ \hat{w}_{n1}/\hat{w}_{n*}\hat{w}_{*1} & \hat{w}_{n2}/\hat{w}_{n*}\hat{w}_{*2} & \cdots & \hat{w}_{nn}/\hat{w}_{n*}\hat{w}_{*n} \end{bmatrix} \begin{bmatrix} \hat{w}_{*1} \\ \hat{w}_{*2} \\ \vdots \\ \hat{w}_{n*} \end{bmatrix} = n \begin{bmatrix} \hat{w}_1/\hat{w}_{1*} \\ \hat{w}_2/\hat{w}_{2*} \\ \vdots \\ \hat{w}_n/\hat{w}_{n*} \end{bmatrix} \tag{8}$$

Equation (8) is of the form

$$\begin{bmatrix} 1 & 1 & . & . & . & 1 \\ 1 & 1 & . & . & . & 1 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 1 & 1 & . & . & . & 1 \end{bmatrix} \begin{bmatrix} \hat{w}_{*1} \\ \hat{w}_{*2} \\ . \\ . \\ . \\ \hat{w}_{*n} \end{bmatrix} = n \begin{bmatrix} \hat{w}_{**} \\ \hat{w}_{**} \\ . \\ . \\ . \\ \hat{w}_{**} \end{bmatrix} \qquad (9)$$

That is, apart from inconsistencies, the elements of the matrix should approximate to 1. Also, from (3) and (8) the $\hat{w}_{*j}$ are approximately the same and equal to $\hat{w}_{**}$.

The ratio of the underlying weights of objects i and j is now given by:

$$\frac{\hat{w}_{i*}}{\hat{w}_{j*}} = \frac{\left[\prod_{k=1}^{n} \frac{\hat{w}_{ik}}{\hat{w}_{*k}}\right]^{\frac{1}{n}}}{\left[\prod_{k=1}^{n} \frac{\hat{w}_{jk}}{\hat{w}_{*k}}\right]^{\frac{1}{n}}} = \left[\prod_{k=1}^{n} \frac{\hat{w}_{ik}}{\hat{w}_{jk}}\right]^{\frac{1}{n}} = \left[\prod_{k=1}^{n} \frac{a_{ik}}{a_{jk}}\right]^{\frac{1}{n}} \qquad (10)$$

Thus multiplicative synthesisation enables one to extract estimates of ratios of underlying weights of any pair of objects from the relevant two rows only in which they are involved in pairwise comparisons.

$$\text{Let} \quad \bar{a}_{ij} = \frac{\hat{w}_{i*}}{\hat{w}_{j*}} \quad \text{for } i, j = 1, 2, ..., n \qquad (11)$$

The $\bar{a}_{ij}$ represent the ratio of elements of row i to the elements of row j in a consistent weight ratio matrix corresponding to the judgement matrix $\mathbf{A} = [a_{ij}]$. Since all elements in the main diagonal of the consistent weight ratio matrix are unity, the elements above the main diagonal will have values of $\bar{a}_{i\,i+1}$ for i = 1, 2, ..., n-1. The consistent weight ratio matrix will look as follows:

$$\begin{bmatrix} 1 & \bar{a}_{12} & & & & \\ & 1 & \bar{a}_{23} & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot\ \cdot & \\ & & & & \cdot & \bar{a}_{(n-1)n} \\ & & & & & 1 \end{bmatrix} \qquad (12)$$

Because the weight ratio matrix is consistent the rest of the entries of this matrix are filled by using the following consistency condition:

$$a_{ij} = a_{ik} * a_{kj}, \text{ for each } i, j, k = 1, 2, \ldots, n.$$

together with the reciprocal property. For example,

$$\bar{a}_{13} = \frac{\hat{w}_{1*}}{\hat{w}_{3*}} = \frac{\hat{w}_{1*}}{\hat{w}_{2*}} \times \frac{\hat{w}_{2*}}{\hat{w}_{3*}} = \bar{a}_{12} \cdot \bar{a}_{23}$$

Consequently, once the synthesised weights $\bar{a}_{i\,i+1}$ have been found the rest of the matrix is fully determined. So the n-1 elements above the main diagonal contain all judgement information synthesised into comparisons between rows.

The result of this procedure is a consistent weight ratio matrix whose normalised columns yield the underlying ratio scale of entities. The advantage of this procedure is its simplicity of use.

Interpretation of Pairwise Judgements.

It has been shown that $\bar{a}_{12}$ is a synthesis of the ratios of weights w_1 to w_2 as reflected in the ratio of the elements of row one to those of row two. Thus, for column two, the ratio is $\bar{a}_{12}$ to 1 and, for column 3, it is $\bar{a}_{13}$ to $\bar{a}_{23}$, which is $\frac{\bar{a}_{13}}{\bar{a}_{23}} = \frac{\bar{a}_{12}}{1} = \bar{a}_{12}$. In fact all corresponding elements in row one and row two in the revised and now consistent matrix will have the same ratio: $\bar{a}_{12}$.

This throws some light on original pairwise judgements which provided the data from which $\bar{a}_{12}$ was synthesised. If the respondent had made these judgements perfectly consistently then each would produce the same ratio $\bar{a}_{12}$. Consider the following judgement matrix

$$\begin{bmatrix} 1 & \frac{1}{7} & 3 & 5 \\ 7 & 1 & 8 & 9 \\ \frac{1}{3} & \frac{1}{8} & 1 & 3 \\ \frac{1}{5} & \frac{1}{9} & \frac{1}{3} & 1 \end{bmatrix} \quad (13)$$

For the ratio between rows one and two, columns one and two both provide a direct ratio one with two and two with one. Rows one and two are also compared indirectly in column three by means of their relationship through weight three, and similarly for column four.

The implication is that, if the issue on which weight three is based is independent of those on which weights one and two are based, then the weight ratio between one and two should be the same whether it is determined directly or indirectly through the medium of another weight. This point has important implications with regard to reducing inconsistency.

A further consequence of this interpretation is the modification of equation (10). For any pair of rows the synthesis of the weight ratios should not take all n elements of the two rows, but only the n-1 judgements affecting comparisons between these rows. As it is, comparisons between row i and row j produce double counting: i with j and j with i. Hence equations (10) should be modified to read

$$\bar{a}_{ij} = \frac{\hat{w}_{i*}}{\hat{w}_{j*}} = \left[\prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{a_{ik}}{a_{jk}} \right]^{\frac{1}{n-1}} \quad (14)$$

Using example (13) above we would get

$$\bar{a}_{34} = \left[\frac{1/3}{1/5} \text{ x } \frac{1/8}{1/9} \text{ x } \frac{3}{1} \right]^{\frac{1}{3}} = \left[\frac{5}{3} \text{ x } \frac{9}{8} \text{ x } \frac{3}{1} \right]^{\frac{1}{3}} = 1.778$$

Worked Example.

The application of the revised procedure using (14) to the judgement matrix (13) yields the following weight ratio consistent matrix:

$$\begin{bmatrix} 1 & 0.310 & 2.467 & 4.385 \\ 3.226 & 1 & 7.958 & 14.153 \\ 0.405 & 0.126 & 1 & 1.788 \\ 0.228 & 0.071 & 0.562 & 1 \end{bmatrix} \quad (15)$$

The ratios of all the values between any two rows will be the same. From any row the weights of all the others can be read directly. For instance if object 2 is given a weight of 1, then object 1 is, on average, 3.226 times as important, object 3 is 7.958 times and object 4 is 14.153 times as important. Any normalisation procedure should produce the same values in each row and the ratio between adjoining pairs of rows will correspond to the $\bar{a}_{i\,i+1}$.

To get weight values which have a geometric mean of one, take the geometric mean of the first row values giving:

$$w_1 = [0.310 \text{ x } 2.467 \text{ x } 4.385]^{\frac{1}{4}} = 1.353$$

Since $\dfrac{w_1}{w_2} = \bar{a}_{12}$ it follows that $w_2 = \dfrac{w_1}{\bar{a}_{12}} = \dfrac{1.353}{0.310} = 4.365$

Similarly $w_3 = \dfrac{w_2}{\bar{a}_{23}} = 0.548$ and $w_4 = \dfrac{w_3}{\bar{a}_{34}} = 0.309$

Hence the underlying ratio scale of entities whose inconsistent judgement matrix is (13) is:

$$\mathbf{u} = (1.353,\ 4.365,\ 0.548,\ 0.309)$$

Note that $u_i > 0$ and $\prod_{i=1}^{n} u_i = 1$ for i = 1, 2, ..., n.

Summary

In this paper a procedure for extracting consistent weight ratio matrices from inconsistent judgement matrices has been proposed. The advantages of this method include its ease of application, its freedom from inverse inconsistency which affects the eigenvector method and it has potential of avoiding rank reversals.

Further research is needed to investigate the effectiveness of the proposed method in discriminating between consistent and inconsistent judgement matrices.

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